Stratifications of Abelian Categories

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Programme

1. Motivation
   - Playtime
2. Recollements of Abelian Categories
   - Definitions
   - Properties
3. Stratifications of Abelian Categories
   - Definitions
   - Stratifications of Module Categories
   - Highest Weight Categories

Stratification of Topological Spaces

Stratification of a topological space

A **stratification of a topological space** X consists of a finite collection

$$\{X_\lambda\}_{\lambda \in \Lambda}$$

of disjoint, connected, locally closed subspaces (called *strata*), in which

- $X = \bigcup_{\lambda \in \Lambda} X_\lambda$
- Each $\overline{X_\lambda}$ is a union of strata.

NB. Associated to such a stratification is the poset $(\Lambda, \leq)$ defined

$$\mu \leq \lambda \text{ if } X_\mu \subset \overline{X_\lambda}.$$

Aim

- Construct an analogue of this construction for abelian categories i.e **define a stratification of an abelian category by a poset** $(\Lambda, \leq)$.
- Need: Analogue of “attaching" two abelian categories. This will be achieved by **recollements**.

History

1982 Beilinson, Bernstein, Deligne: *Faisceaux pervers*

- This paper defines recollements of triangulated categories.
- Recollements of abelian categories are an abelian category analogue of this definition.

1988 Cline, Parshall, Scott: *Finite dimensional categories and highest weight categories*

1996 Cline, Parshall, Scott: *Stratifying endomorphism algebras*

1998 Agoston, Dlab, Lukacs: *Stratified algebras*

2018 Brundan, Stroppel: *Semi-infinite highest weight categories*

Abelian categories

Let:

- $\Bbbk$ be a field.
- $\mathcal{A}$ be a $\Bbbk$-linear abelian category.

Recall:

- An object $L \in \mathcal{A}$ is **simple** if L has no subobjects.
- Let $\mathcal{C}$ be a class of objects in $\mathcal{A}$ (closed under isomorphisms). A **finite filtration** of an object $X \in \mathcal{A}$ by objects in $\mathcal{C}$ consists of a chain of monomorphisms

$$0 = X_0 \hookrightarrow X_1 \hookrightarrow \cdots \hookrightarrow X_n = X$$

  in which $X_i/X_{i-1} \in \mathcal{C}$ for each i.
- A **filtration** of X by objects in $\mathcal{C}$ consists of a (possibly infinite) chain of monomorphisms

$$0 = X_0 \hookrightarrow X_1 \hookrightarrow \cdots \hookrightarrow X_n \hookrightarrow \cdots$$

  in which $X = \operatorname{colim} X_i$ and each $X_i/X_{i-1} \in \mathcal{C}$.

Playtime

Desiderata: What features would we want the “attaching" of two abelian categories to satisfy?

Let $\mathcal{A}_Z$ and $\mathcal{A}_U$ be abelian categories.
We want to study abelian categories $\mathcal{A}$ in which:

- There are fully faithful functors:

$$\mathcal{A}_Z \xhookrightarrow{i_*} \mathcal{A} \xhookleftarrow{j_{!*}} \mathcal{A}_U$$

- The essential image of i_* does not intersect the essential image of $j_{!*}$.
- Every object $X \in \mathcal{A}$ has a filtration by objects in $\operatorname{im} i_* \cup \operatorname{im} j_{!*}$.
- The essential image of i_* is a Serre subcategory of $\mathcal{A}$.

Playtime

Slightly Stronger Desiderata

Let $\mathcal{A}_Z$ and $\mathcal{A}_U$ be abelian categories.
We want to study abelian categories $\mathcal{A}$ in which:

- There are fully faithful functors:

$$\mathcal{A}_Z \xhookrightarrow{i_*} \mathcal{A} \xhookleftarrow{j_{!*}} \mathcal{A}_U$$

- Every simple object $L \in \mathcal{A}$ is uniquely either of the form $i_* L'$ (for $L' \in \mathcal{A}_Z$) or $j_{!*} L'$ (for $L' \in \mathcal{A}_U$).
- The essential image of i_* is a Serre subcategory of $\mathcal{A}$.

Playtime: The case $\mathcal{A}_Z \simeq \mathcal{A}_U \simeq \mathbb{k}\text{-mod}$

The case $\mathcal{A}_Z \simeq \mathcal{A}_U \simeq \mathbb{k}\text{-mod}$

What categories $\mathcal{A}$ satisfy:

- There are fully faithful functors

$$\mathbb{k}\text{-mod} \xhookrightarrow{i_*} \mathcal{A} \xhookleftarrow{j_{!*}} \mathbb{k}\text{-mod}$$

  with disjoint essential images.
- Every object $X \in \mathcal{A}$ has a filtration by objects in $\operatorname{im} i_* \cup \operatorname{im} j_{!*}$.

Playtime: The case $\mathcal{A}_Z \simeq \mathcal{A}_U \simeq \Bbbk\text{-mod}$

$\Bbbk\text{-mod} \oplus \Bbbk\text{-mod}$

Consider the direct sum category $\Bbbk\text{-mod} \oplus \Bbbk\text{-mod}$:

- There are functors:

$$
\begin{aligned}
i_* &: \Bbbk\text{-mod} \to \Bbbk\text{-mod} \oplus \Bbbk\text{-mod}; & V \mapsto (V, 0) \\
j_{!*} &: \Bbbk\text{-mod} \to \Bbbk\text{-mod} \oplus \Bbbk\text{-mod}; & V \mapsto (0, V)
\end{aligned}
$$

- The simple objects of $\Bbbk\text{-mod} \oplus \Bbbk\text{-mod}$ are $(\Bbbk, 0)$ and $(0, \Bbbk)$.
- Every object $(V, W) \in \Bbbk\text{-mod} \oplus \Bbbk\text{-mod}$ has a filtration

$$(0, 0) \hookrightarrow (V, 0) \hookrightarrow (V, W)$$

Playtime: The case $\mathcal{A}_Z \simeq \mathcal{A}_U \simeq \Bbbk\text{-mod}$

$(\bullet \to \bullet)\text{-mod}$

Consider the category, $(\bullet \to \bullet)\text{-mod}$, of functors $\{x \to y\} \to \Bbbk\text{-mod}$. Alternatively,

- *Objects*: Linear maps $V \xrightarrow{f} W$
- *Morphisms*: Commutative diagrams

$$\begin{array}{ccc} V & \xrightarrow{\varphi_V} & V' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ W & \xrightarrow[\varphi_W]{} & W' \end{array}$$

Playtime: The case $\mathcal{A}_Z \simeq \mathcal{A}_U \simeq \Bbbk\text{-mod}$

$(\bullet \to \bullet)\text{-mod}$

- There are functors:

$$\begin{aligned} i_* &: \Bbbk\text{-mod} \to (\bullet \to \bullet)\text{-mod}; \qquad && V \mapsto (V \to 0) \\ j_{!*} &: \Bbbk\text{-mod} \to (\bullet \to \bullet)\text{-mod}; \qquad && V \mapsto (0 \to V) \end{aligned}$$

- The simple objects of $(\bullet \to \bullet)\text{-mod}$ are $(\Bbbk \to 0)$ and $(0 \to \Bbbk)$.
- Every object $(V \xrightarrow{f} W) \in (\bullet \to \bullet)\text{-mod}$ has a filtration

$$\begin{array}{ccccc} 0 & \hookrightarrow & 0 & \hookrightarrow & V \\ \downarrow & & \downarrow & & \downarrow f \\ 0 & \hookrightarrow & W & \hookrightarrow & W \end{array}$$

Recollement of Abelian Categories

A **recollement** of $\mathcal{A}$ consists of categories and functors:

$$\begin{array}{ccccc} & \xleftarrow{\;i^*\;} & & \xleftarrow{\;j_!\;} & \\ \mathcal{A}_Z & \xrightarrow{\;i\;} & \mathcal{A} & \xrightarrow{\;j\;} & \mathcal{A}_U \\ & \xleftarrow[\;i^!\;]{} & & \xleftarrow[\;j_*\;]{} & \end{array}$$

satisfying the conditions:

(R1) $(i^*, i, i^!)$ and $(j_!, j, j_*)$ are adjoint triples.

(R2) The functors i, $j_!$, j_* are fully-faithful.

(R3) The functors satisfy $j \circ i = 0$ (and so by adjunction $i^* j_! = 0 = i^! j_*$).

(R4) For any object $X \in \mathcal{A}$, if $j(X) = 0$ then X is in the essential image of i.

Examples: Recollement of Abelian Categories

Direct sum of categories

$$\mathcal{A} \underset{\xleftarrow[i^!]{}}{\overset{\xleftarrow{i^*}}{\xrightarrow{i}}} \mathcal{A} \oplus \mathcal{B} \underset{\xleftarrow[j_*]{}}{\overset{\xleftarrow{j_!}}{\xrightarrow{j}}} \mathcal{B}$$

where i and $j_! = j_*$ are the inclusion functors, and $i^* = i^!$ and j are projection functors.

Examples: Recollement of Abelian Categories

$\bullet \to \bullet$ Quiver Representations

$$
\Bbbk\text{-mod} \;\underset{\xleftarrow[i^!]{}}{\overset{\xleftarrow{i^*}}{\xrightarrow{i}}}\; (\bullet \to \bullet)\text{-mod} \;\underset{\xleftarrow[j_*]{}}{\overset{\xleftarrow{j_!}}{\xrightarrow{j}}}\; \Bbbk\text{-mod}
$$

where:

$$
\begin{aligned}
i &: \Bbbk\text{-mod} \to A_2\text{-mod}; && V \mapsto (V \to 0) \\
j &: A_2\text{-mod} \to \Bbbk\text{-mod}; && (V \to W) \mapsto W \\
i^* &: A_2\text{-mod} \to \Bbbk\text{-mod}; && (V \to W) \mapsto V \\
i^! &: A_2\text{-mod} \to \Bbbk\text{-mod}; && (V \xrightarrow{f} W) \mapsto \ker f \\
j_! &: \Bbbk\text{-mod} \to A_2\text{-mod}; && V \mapsto (0 \to V) \\
j_* &: \Bbbk\text{-mod} \to A_2\text{-mod}; && V \mapsto (V \to V)
\end{aligned}
$$

Examples: Recollement of Abelian Categories

$\bullet \to \bullet$ Quiver Representations (2)

$$\mathbb{k}\text{-mod} \;\underset{i^!}{\overset{\overset{i^*}{\longleftarrow}}{\underset{\longleftarrow}{\xrightarrow{i}}}}\; (\bullet \to \bullet)\text{-mod} \;\underset{j_*}{\overset{\overset{j_!}{\longleftarrow}}{\underset{\longleftarrow}{\xrightarrow{j}}}}\; \mathbb{k}\text{-mod}$$

where:

$$\begin{aligned}
i &: \mathbb{k}\text{-mod} \to A_2\text{-mod}; & V &\mapsto (0 \to V) \\
j &: A_2\text{-mod} \to \mathbb{k}\text{-mod}; & (V \to W) &\mapsto V \\
i^* &: A_2\text{-mod} \to \mathbb{k}\text{-mod}; & (V \xrightarrow{f} W) &\mapsto \operatorname{cok} f \\
i^! &: A_2\text{-mod} \to \mathbb{k}\text{-mod}; & (V \to W) &\mapsto W \\
j_! &: \mathbb{k}\text{-mod} \to A_2\text{-mod}; & V &\mapsto (V \to V) \\
j_* &: \mathbb{k}\text{-mod} \to A_2\text{-mod}; & V &\mapsto (V \to 0)
\end{aligned}$$

Examples: Recollement of Abelian Categories

$\bullet \to \bullet \to \bullet$ Quiver Representations

$$(\bullet \to \bullet)\text{-mod} \;\underset{i^!}{\overset{i^*}{\leftarrow}}\!\!\xrightarrow{\;i\;}\; (\bullet \to \bullet \to \bullet)\text{-mod} \;\underset{j_*}{\overset{j_!}{\leftarrow}}\!\!\xrightarrow{\;j\;}\; \Bbbk\text{-mod}$$

where:

$$i : A_2\text{-mod} \to A_3\text{-mod}; \qquad (U \to V) \mapsto (U \to V \to 0)$$

$$j : A_3\text{-mod} \to \Bbbk\text{-mod}; \qquad (U \to V \to W) \mapsto W$$

Examples: Recollement of Abelian Categories

Λ-Quiver Representations

- Let Λ be a poset, thought of as a quiver.
- Let $\lambda \in \Lambda$ be maximal.

There is a recollement

$$\Lambda \setminus \{\lambda\}\text{-mod} \; \begin{smallmatrix} \xleftarrow{i^*} \\ \xrightarrow{i} \\ \xleftarrow[i^!]{} \end{smallmatrix} \; \Lambda\text{-mod} \; \begin{smallmatrix} \xleftarrow{j_!} \\ \xrightarrow{j} \\ \xleftarrow[j_*]{} \end{smallmatrix} \; \mathbb{k}\text{-mod}$$

Examples: Recollement of Abelian Categories

Constructible Sheaves

$$\mathcal{S}h(Z,\Bbbk) \underset{{}^0 i^!}{\overset{\overset{{}^0 i^*}{\longleftarrow}}{\underset{\longleftarrow}{\xrightarrow{\;i\;}}}} \mathcal{S}h(X,\Bbbk) \underset{{}^0 j_*}{\overset{\overset{{}^0 j_!}{\longleftarrow}}{\underset{\longleftarrow}{\xrightarrow{\;j\;}}}} \mathcal{S}h(U,\Bbbk)$$

Properties of Recollements

Consider a recollement:

$$\mathcal{A}_Z \;\substack{\xleftarrow{i^*} \\ \xrightarrow{i} \\ \xleftarrow[i^!]{}}\; \mathcal{A} \;\substack{\xleftarrow{j_!} \\ \xrightarrow{j} \\ \xleftarrow[j_*]{}}\; \mathcal{A}_U$$

- The essential image of i is a Serre subcategory of $\mathcal{A}$.
- The adjunction maps

$$i^* \circ i \to \mathrm{Id} \to i^! \circ i$$
$$j \circ j_* \to \mathrm{Id} \to j \circ j_!$$

  are isomorphisms.

(This is equivalent to $i, j_!, j_*$ being fully-faithful)

Properties of Recollements

Consider a recollement:

$$\mathcal{A}_Z \underset{i^!}{\overset{i^*}{\leftarrow}} \xrightarrow{i} \mathcal{A} \underset{j_*}{\overset{j_!}{\leftarrow}} \xrightarrow{j} \mathcal{A}_U$$

For object $X \in \mathcal{A}$:

- $i \circ i^*(X)$ is the largest quotient of $\mathcal{A}$ in the essential image of i.
- $i \circ i^!(X)$ is the largest subobject of $\mathcal{A}$ in the essential image of i.

Theorem

Let $\mathcal{A}^U$ be the full subcategory of $\mathcal{A}$ whose objects have no quotients or subobjects in the essential image of i.
The functor j restricts to an equivalence of categories

$$j : \mathcal{A}^U \to \mathcal{A}_U$$

$(\bullet \to \bullet)$-mod

Consider the recollement:

$$\mathbb{k}\text{-mod} \; \underset{i^!}{\overset{i^*}{\underset{\longleftarrow}{\overset{\longleftarrow}{\xrightarrow{\;i\;}}}}} \; (\bullet \to \bullet)\text{-mod} \; \underset{j_*}{\overset{j_!}{\underset{\longleftarrow}{\overset{\longleftarrow}{\xrightarrow{\;j\;}}}}} \; \mathbb{k}\text{-mod}$$

where:

$$\begin{aligned} i &: \mathbb{k}\text{-mod} \to A_2\text{-mod}; & V &\mapsto (V \to 0) \\ j &: A_2\text{-mod} \to \mathbb{k}\text{-mod}; & (V \to W) &\mapsto W \end{aligned}$$

Then $\mathcal{A}^U \simeq \{0 \to W\}$. The functor $j : \mathcal{A}^U \to \mathbb{k}\text{-mod}$ has quasi-inverse $j_!$.

Examples

$(\bullet \to \bullet)$-mod

Consider the recollement:

$$\mathbb{k}\text{-mod} \;\underset{i^!}{\overset{\overset{i^*}{\longleftarrow}}{\underset{\longleftarrow}{\xrightarrow{\;i\;}}}}\; (\bullet \to \bullet)\text{-mod} \;\underset{j_*}{\overset{\overset{j_!}{\longleftarrow}}{\underset{\longleftarrow}{\xrightarrow{\;j\;}}}}\; \mathbb{k}\text{-mod}$$

where:

$$i : \mathbb{k}\text{-mod} \to A_2\text{-mod}; \qquad V \mapsto (0 \to V)$$
$$j : A_2\text{-mod} \to \mathbb{k}\text{-mod}; \qquad (V \to W) \mapsto V$$

Then $\mathcal{A}^U \simeq \{V \to 0\}$. The functor $j : \mathcal{A}^U \to \mathbb{k}\text{-mod}$ has quasi-inverse j_*.

Intermediate Extension Functor

Consider a recollement:

$$\mathcal{A}_Z \underset{\overset{\longleftarrow}{i^!}}{\overset{\overset{i^*}{\longleftarrow}}{\xrightarrow{\;i\;}}} \mathcal{A} \underset{\overset{\longleftarrow}{j_*}}{\overset{\overset{j_!}{\longleftarrow}}{\xrightarrow{\;j\;}}} \mathcal{A}_U$$

Question

What is the quasi-inverse of the equivalence $j : \mathcal{A}^U \to \mathcal{A}_U$?

Intermediate Extension Functor

Consider a recollement:

$$\mathcal{A}_Z \; \underset{i^!}{\overset{i^*}{\leftarrow}} \xrightarrow{i} \; \mathcal{A} \; \underset{j_*}{\overset{j_!}{\leftarrow}} \xrightarrow{j} \; \mathcal{A}_U$$

Consider the isomorphisms:

$$\mathrm{Hom}_{\mathcal{A}}(j_! X, j_* X) \simeq \mathrm{Hom}_{\mathcal{A}_U}(X, j \circ j_* X) \simeq \mathrm{Hom}_{\mathcal{A}_U}(X, X).$$

Let $\overline{1_X} : j_! X \to j_* X$ be the morphism corresponding to $1_X : X \to X$.

Intermediate Extension Functor

Define the functor $j_{!*} : \mathcal{A}_U \to \mathcal{A}$:

$$j_{!*} X := \mathrm{im}(\overline{1_X} : j_! X \to j_* X) \in \mathcal{A}.$$

The image of $j_{!*}$ is in $\mathcal{A}^U$ and $j_{!*} : \mathcal{A}_U \to \mathcal{A}^U$ is quasi-inverse to $j : \mathcal{A}^U \to \mathcal{A}_U$.

Examples: Intermediate Extension Functor

Consider the recollement:

$$\mathbb{k}\text{-mod} \;\; \begin{array}{c} \xleftarrow{\quad i^* \quad} \\ \xrightarrow{\quad i \quad} \\ \xleftarrow[\quad i^! \quad]{} \end{array} \;\; (\bullet \to \bullet)\text{-mod} \;\; \begin{array}{c} \xleftarrow{\quad j_! \quad} \\ \xrightarrow{\quad j \quad} \\ \xleftarrow[\quad j_* \quad]{} \end{array} \;\; \mathbb{k}\text{-mod}$$

where:

$$\begin{aligned} i &: \mathbb{k}\text{-mod} \to A_2\text{-mod}; & V &\mapsto (V \to 0) \\ j &: A_2\text{-mod} \to \mathbb{k}\text{-mod}; & (V \to W) &\mapsto W \\ j_! &: \mathbb{k}\text{-mod} \to A_2\text{-mod}; & V &\mapsto (0 \to V) \\ j_* &: \mathbb{k}\text{-mod} \to A_2\text{-mod}; & V &\mapsto (V \to V) \end{aligned}$$

Then

- $\overline{1_V} : j_! V \to j_* V$ is the natural inclusion.
- $j_{!*} : V \mapsto (0 \to V)$ i.e. $j_{!*} = j_!$.

Examples: Intermediate Extension Functor

Consider the recollement:

$$\mathbb{k}\text{-mod} \underset{\xleftarrow[i^!]{}}{\overset{\xleftarrow{i^*}}{\xrightarrow{i}}} (\bullet \to \bullet)\text{-mod} \underset{\xleftarrow[j_*]{}}{\overset{\xleftarrow{j_!}}{\xrightarrow{j}}} \mathbb{k}\text{-mod}$$

where:

$$\begin{aligned} i &: \mathbb{k}\text{-mod} \to A_2\text{-mod}; & W &\mapsto (0 \to W) \\ j &: A_2\text{-mod} \to \mathbb{k}\text{-mod}; & (V \to W) &\mapsto V \\ j_! &: \mathbb{k}\text{-mod} \to A_2\text{-mod}; & V &\mapsto (V \to V) \\ j_* &: \mathbb{k}\text{-mod} \to A_2\text{-mod}; & V &\mapsto (V \to 0) \end{aligned}$$

Then

- $\overline{1_V} : j_! V \to j_* V$ is the natural surjection.
- $j_{!*} : V \mapsto (V \to 0)$ i.e. $j_{!*} = j_*$.

Classification of Simple Objects

Theorem

Consider a recollement:

$$\mathcal{A}_Z \underset{i^!}{\overset{i^*}{\longleftarrow}} \xrightarrow{i} \mathcal{A} \underset{j_*}{\overset{j_!}{\longleftarrow}} \xrightarrow{j} \mathcal{A}_U$$

Every simple object in $\mathcal{A}$ is either of the form

- $i(L)$ for a unique simple object $L \in \mathcal{A}_Z$,

or

- $j_{!*}(L)$ for a unique simple object $L \in \mathcal{A}_U$.

Desiderata Revisited

Desiderata

Let $\mathcal{A}_Z$ and $\mathcal{A}_U$ be abelian categories.
We want to study abelian categories $\mathcal{A}$ in which:

- There are fully faithful functors:

$$\mathcal{A}_Z \xhookrightarrow{i_*} \mathcal{A} \xhookleftarrow{j_{!*}} \mathcal{A}_U$$

- Every simple object $L \in \mathcal{A}$ is uniquely either of the form $i_* L'$ (for $L' \in \mathcal{A}_Z$) or $j_{!*} L'$ (for $L' \in \mathcal{A}_U$).
- The essential image of i_* is a Serre subcategory of $\mathcal{A}$.

Definition: Stratification of Abelian Categories

Stratification of Abelian Categories

A **stratification** of $\mathcal{A}$ by a poset Λ consists of:

- Abelian categories $\mathcal{A}_\lambda$ for each $\lambda \in \Lambda$ (called *strata categories*).
- Serre subcategories $\mathcal{A}_{\Lambda'} \subset \mathcal{A}$ for each downwards-closed subposet of $\Lambda' \subset \Lambda$.

satisfying conditions:

(S1) $\mathcal{A}_\emptyset = 0$, $\mathcal{A}_\Lambda = \mathcal{A}$, and $\Lambda' \subset \Lambda''$ implies $\mathcal{A}_{\Lambda'} \subset \mathcal{A}_{\Lambda''}$.

(S2) For each $\lambda \in \Lambda$ and downwards-closed subset $\Lambda' \subset \Lambda$ in which $\lambda \in \Lambda'$ is maximal, the embedding $i : \mathcal{A}_{\Lambda' \setminus \{\lambda\}} \to \mathcal{A}_{\Lambda'}$ fits into a recollement

$$\mathcal{A}_{\Lambda'\setminus\{\lambda\}} \underset{\xleftarrow{\;i^!\;}}{\overset{\xleftarrow{\;i^*\;}}{\xrightarrow{\;i\;}}} \mathcal{A}_{\Lambda'} \underset{\xleftarrow{\;j_*\;}}{\overset{\xleftarrow{\;j_!\;}}{\xrightarrow{\;j\;}}} \mathcal{A}_\lambda$$

Definition: Stratification of Abelian categories

Stratification by singleton poset: $\{1\}$

$$\mathcal{A}_\emptyset = 0 \quad \underset{\xleftarrow[i^!]{}}{\overset{\xleftarrow{i^*}}{\xrightarrow{i}}} \quad \mathcal{A} = \mathcal{A}_{\{1\}} \quad \underset{\xleftarrow[j_*]{}}{\overset{\xleftarrow{j_!}}{\xrightarrow{j}}} \quad \mathcal{A}_1$$

This is the data of an autoequivalence $j : \mathcal{A} \to \mathcal{A}$.

Stratification by poset: $1 \leq 2$

$$\mathcal{A}_1 \simeq \mathcal{A}_{\{1\}} \quad \underset{\xleftarrow[i^!]{}}{\overset{\xleftarrow{i^*}}{\xrightarrow{i}}} \quad \mathcal{A} = \mathcal{A}_{\{1,2\}} \quad \underset{\xleftarrow[j_*]{}}{\overset{\xleftarrow{j_!}}{\xrightarrow{j}}} \quad \mathcal{A}_2$$

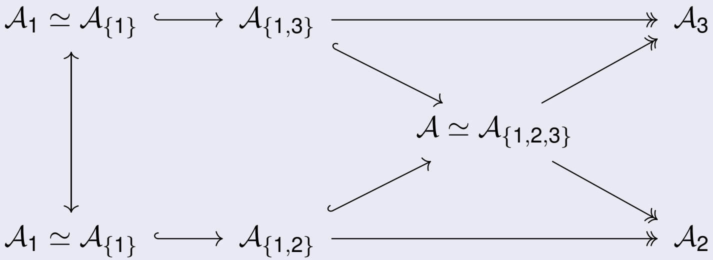
Definition: Stratification of Abelian Categories

Stratification by poset: $1 \leq 2 \leq 3$

$$
\begin{array}{ccccc}
\mathcal{A}_{\{1\}} \simeq \mathcal{A}_1 & & & & \\
\Big\downarrow & & & & \\
\mathcal{A}_{\{1,2\}} & \hookrightarrow & \mathcal{A} \simeq \mathcal{A}_{\{1,2,3\}} & \twoheadrightarrow & \mathcal{A}_3 \\
\Big\downarrow & & & & \\
\mathcal{A}_2 & & & &
\end{array}
$$

Definition: Stratification of Abelian Categories

Stratification by poset: $2 \geq 1 \leq 3$



Classification of Simple Objects

Let $\mathcal{A}$ have a stratification by a poset Λ.

- For $\lambda \in \Lambda$, define the functors

$$j_*^\lambda, j_!^\lambda, j_{!*}^\lambda : \mathcal{A}_\lambda \to \mathcal{A}_{\{\mu\in\Lambda \mid \mu\leq\lambda\}} \hookrightarrow \mathcal{A}$$

- For each simple object $L \in \mathcal{A}$, there is a unique $\lambda \in \Lambda$ and unique simple object $L_\lambda \in \mathcal{A}_\lambda$ in which $L = j_{!*}^\lambda L_\lambda$.

Preorder on simple objects

Let $\{L(b)\}_{b\in B}$ be the set of simple objects in $\mathcal{A}$ (up to isomorphism). There is a preorder $\lesssim$ on B defined:

- If $L(b_1) = j_{!*}^\mu L_\mu(b_1)$ and $L(b_2) = j_{!*}^\lambda L_\lambda(b_2)$, then

$$b_1 \lesssim b_2 \qquad \text{if } \mu \leq \lambda.$$

Stratifications with enough projectives

Theorem (original - see my thesis)

A $\Bbbk$-linear abelian category with a stratification is equivalent to a category of finite dimensional modules of a finite dimensional $\Bbbk$-algebra if and only if the same is true for all strata categories.

Proof Sketch

- For each simple object $L(b) \in \mathcal{A}$ construct a projective cover $P(b) \twoheadrightarrow L(b)$.
- Then $\mathcal{A} \simeq \mathrm{End}(\bigoplus_{b \in B} P(b))^{op}\text{-mod}$.

Example: Stratifications with enough projectives

$(\bullet \to \bullet)$-mod

Simple objects: $L_1 = 0 \to \mathbb{k}$ and $L_2 = \mathbb{k} \to 0$

Projective indecomposables: $P_1 = 0 \to \mathbb{k}$ and $P_2 = \mathbb{k} \to \mathbb{k}$

Injective indecomposables: $I_1 = \mathbb{k} \to \mathbb{k}$ and $I_2 = \mathbb{k} \to 0$

Standard and Costandard Objects

How to construct the projective cover $P(b) \twoheadrightarrow L(b)$

- Let $L(b) = j^{\lambda}_{!*} L_{\lambda}(b)$.
- Let $P_{\lambda}(b) \twoheadrightarrow L_{\lambda}(b)$ be the projective cover in $\mathcal{A}_{\lambda}$.
- Let $\Delta(b) = j^{\lambda}_{!} P_{\lambda}(b)$.
  Note that $\Delta(b)$ is indecomposable and $\Delta(b) \twoheadrightarrow L(b)$ but $\Delta(b)$ is not necessarily projective.
- We prove the existence of a projective indecomposable object $P(b)$ and surjection

$$P(b) \twoheadrightarrow \Delta(b) \twoheadrightarrow L(b).$$

- Call the objects $\Delta(b)$ **standard objects.**
- Dually, define **costandard objects** $\nabla(b)$ using injective envelopes.

Standard and Costandard Objects

Theorem (original - see my thesis)

For each $b \in B$:

(i) The projective cover, $P(b)$, of $L(b)$ fits into a short exact sequence

$$0 \to Q(b) \to P(b) \to \Delta(b) \to 0$$

in which $Q(b)$ has a filtration by quotients of $\Delta(b')$ satisfying $b' > b$.

(ii) The injective envelope, $I(b)$, of $L(b)$ fits into a short exact sequence

$$0 \to \nabla(b) \to I(b) \to Q'(b) \to 0$$

in which $Q'(b)$ has a filtration by subobjects of $\nabla(b')$ satisfying $b' > b$.

Standard and Costandard objects

Question

(i) Under what conditions do projective objects have a filtration by standard objects.

(ii) Under what conditions do injective objects have a filtration by costandard objects.

Application

Tilting theory

Standard and Costandard objects

Homological stratification

A stratification of $\mathcal{A}$ by a poset Λ is **homological** if each inclusion functor $i : \mathcal{A}_{\Lambda'} \hookrightarrow \mathcal{A}$ ($\Lambda' \subset \Lambda$) satisfies

$$\mathrm{Ext}^k_{\mathcal{A}}(X, Y) = \mathrm{Ext}^k_{\mathcal{A}}(i(X), i(Y)) \qquad \text{for all } k \in \mathbb{N}.$$

Theorem (original/unpublished)

Let $\mathcal{A}$ be a finite abelian category with a stratification by a poset Λ.

(i) All projective objects in $\mathcal{A}$ have a filtration by standard objects iff $\mathcal{A}$ has a homological stratification and each $j^{\lambda}_{*} : \mathcal{A}_{\lambda} \to \mathcal{A}$ is exact.

(ii) All injective objects in $\mathcal{A}$ have a filtration by costandard objects iff $\mathcal{A}$ has a homological stratification and each $j^{\lambda}_{!} : \mathcal{A}_{\lambda} \to \mathcal{A}$ is exact.

A more general result can be stated in terms of Brundan and Stroppels ε-stratified categories.

Special case: Highest Weight Categories

Let Λ be a poset.

Definition: Highest Weight Category (Cline, Parshall, Scott)

Suppose $\mathcal{A} \simeq A\text{-mod}$ has simple objects $\{L_\lambda\}_{\lambda \in \Lambda}$.
The category $\mathcal{A}$ is a **highest weight category with respect to** Λ if there are objects $\{\Delta_\lambda\}_{\lambda \in \Lambda}$ satisfying

1. Each Δ_λ fits into a short exact sequence

$$0 \to K_\lambda \to \Delta_\lambda \to L_\lambda \to 0$$

   in which K_λ has a filtration by simple objects $L(\mu)$ in which $\mu < \lambda$.

2. The projective cover, P_λ, of L_λ fits into a short exact sequence

$$0 \to U_\lambda \to P_\lambda \to \Delta_\lambda \to 0$$

   in which U_λ has a filtration by objects Δ_μ with $\mu > \lambda$.

Special case: Highest Weight Categories

Theorem (Krause, 2017: Highest weight categories and recollements)

A $\mathbb{k}$-linear abelian category $\mathcal{A}$ is a highest weight category with respect to the poset Λ if and only if $\mathcal{A}$ has a homological stratification by Λ in which every strata category has a unique simple object.