THE UNIVERSITY OF SYDNEY

Heisenberg Categorification in Positive Characteristic

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Plan

1. The story in characteristic zero (due mainly to Khovanov [2014] and Brundan-Savage-Webster [2019])

2. New Results: A characteristic free approach (due to Hong-W.Yacobi - unpublished)

Khovanov's original result [2014]

Let $\Bbbk$ be an algebraically closed field of characteristic zero.

- Consider the category
$$\mathcal{R} := \bigoplus_{d \in \mathbb{N}} \Bbbk\mathfrak{S}_d\mathrm{Mod}$$
- Recall that
$$K_0(\mathcal{R}) \simeq \mathrm{Sym} := \lim_{n\to\infty} \Bbbk[x_1, \dots, x_n]^{\mathfrak{S}_n}$$
- Consider the *Heisenberg ring* $\mathrm{Heis} \hookrightarrow \mathsf{End}(\mathrm{Sym}) = \mathsf{End}(K_0(\mathcal{R}))$.
- **Khovanov** [2014]: There is a category $\mathcal{H}eis$ and functor $\mathcal{H}eis \to \mathcal{E}nd_{exact}(\mathcal{R})$ making the following commute:

$$\begin{array}{ccc} \mathcal{H}eis & \longrightarrow & \mathcal{E}nd_{exact}(\mathcal{R}) \\ {\scriptstyle K_0(Kar(-))} \downarrow & & \downarrow {\scriptstyle K_0(-)} \\ K_0(Kar(\mathcal{H}eis)) \hookrightarrow \mathrm{Heis} & \hookrightarrow & \mathsf{End}(\mathrm{Sym}) \simeq \mathsf{End}(K_0(\mathcal{R})) \end{array}$$

The category $\mathcal{R} := \bigoplus_{d \in \mathbb{N}} \Bbbk\mathfrak{S}_d\mathrm{Mod}$

Aim: Define a monoidal structure on $\mathcal{R}$

$Ind_{m,n}(-,-) : \Bbbk\mathfrak{S}_m\mathrm{Mod} \times \Bbbk\mathfrak{S}_n\mathrm{Mod} \to \Bbbk\mathfrak{S}_{m+n}\mathrm{Mod}$

- $Ind_{m,n}(M, N) := \Bbbk\mathfrak{S}_{m+n} \otimes_{\Bbbk\mathfrak{S}_m \times \Bbbk\mathfrak{S}_n} (M \boxtimes N)$
- E.g.(1): $Ind_{m,n}(\Bbbk\mathfrak{S}_m, \Bbbk\mathfrak{S}_n) = \Bbbk\mathfrak{S}_{m+n}$
- E.g.(2):
  $Ind_{m,n}(-, \Bbbk\mathfrak{S}_n) = \Bbbk\mathfrak{S}_{m+n} \otimes_{\Bbbk\mathfrak{S}_m} - : \Bbbk\mathfrak{S}_m\mathrm{Mod} \to \Bbbk\mathfrak{S}_{m+n}\mathrm{Mod}$

Induction Product $- \otimes - : \mathcal{R} \times \mathcal{R} \to \mathcal{R}$

- $\mathcal{R}$ has a tensor product:

$$- \otimes - := \bigoplus_{m,n} Ind_{m,n}(-,-) : \mathcal{R} \times \mathcal{R} \to \mathcal{R}$$

The category $\mathcal{R} := \bigoplus_{d\in\mathbb{N}} \Bbbk\mathfrak{S}_d\mathrm{Mod}$

Aim: Define an internal hom on $\mathcal{R}$

$[-,-]_{m,n} : (\Bbbk\mathfrak{S}_m\mathrm{Mod})^{op} \times \Bbbk\mathfrak{S}_n\mathrm{Mod} \to \Bbbk\mathfrak{S}_{n-m}\mathrm{Mod}$ $(n \geqslant m)$

- $[M, N]_{m,n} := \mathrm{Hom}_{\Bbbk\mathfrak{S}_m}(M, N|_{\Bbbk\mathfrak{S}_{\{n-m+1,\dots,n\}}})$
- $\Bbbk\mathfrak{S}_{n-m} \circlearrowright [M, N]_{m,n}$ by postcomposition (since $\Bbbk\mathfrak{S}_{n-m} \hookrightarrow \Bbbk\mathfrak{S}_n \circlearrowright N$)
- E.g. $[\Bbbk\mathfrak{S}_m, -]_{m,n} = Res^{\Bbbk\mathfrak{S}_n}_{\Bbbk\mathfrak{S}_{n-m}}(-) : \Bbbk\mathfrak{S}_n \to \Bbbk\mathfrak{S}_{n-m}$

Internal Hom $[-,-] : \mathcal{R}^{op} \times \mathcal{R} \to \mathcal{R}$

- $\mathcal{R}$ has an internal hom

$$[-,-] := \bigoplus_{m,n} [-,-]_{m,m+n} : \mathcal{R}^{op} \times \mathcal{R} \to \mathcal{R}$$

- i.e. For all objects $M \in \mathcal{R}$, $(- \otimes M, [M, -])$ are an adjoint pair.

The category $\mathcal{R} := \bigoplus_{d \in \mathbb{N}} \Bbbk\mathfrak{S}_d \mathrm{Mod}$

Summary so far:

- $\mathcal{R}$ is a closed symmetric monoidal category i.e. $\mathcal{R}$ has a $\otimes$-product and internal hom $[-,-]$.
- Induction/restriction endofunctors are special cases of applying $\otimes$ and $[-,-]$.
- In particular we have endofunctors

$$\uparrow := - \otimes \Bbbk\mathfrak{S}_1 = \bigoplus_d Ind_{\Bbbk\mathfrak{S}_d}^{\Bbbk\mathfrak{S}_{d+1}}(-) : \mathcal{R} \to \mathcal{R}$$

$$\downarrow := [\Bbbk\mathfrak{S}_1, -] = \bigoplus_d Res_{\Bbbk\mathfrak{S}_d}^{\Bbbk\mathfrak{S}_{d+1}}(-) : \mathcal{R} \to \mathcal{R}$$

and

$$\uparrow^{\circ n} = - \otimes \Bbbk\mathfrak{S}_n : \mathcal{R} \to \mathcal{R}, \qquad \downarrow^{\circ n} = [\Bbbk\mathfrak{S}_n, -] : \mathcal{R} \to \mathcal{R}$$

Recollections about Grothendieck rings

Definition of $K_0(\mathcal{C})$

If $\mathcal{C}$ is a symmetric monoidal abelian category with exact tensor product then $K_0(\mathcal{C})$ is the commutative ring:

- **Generators**: $[X]$ for objects $X \in \mathcal{C}$.
- **Relations**: $[X] + [Z] = [Y]$ whenever there is a short exact sequence:

$$0 \to X \to Y \to Z \to 0$$

- **Multiplication**:

$$[X] \cdot [Y] = [X \otimes Y]$$

$\mathcal{E}nd_{exact}\mathcal{C} \rightsquigarrow \mathrm{End}(K_0(\mathcal{C}))$

If $F : \mathcal{C} \to \mathcal{C}$ is an exact monoidal functor then there is a ring homomorphism $[F] : K_0(\mathcal{C}) \to K_0(\mathcal{C})$:

$$[F] : [X] \mapsto [F(X)]$$

The Grothendieck ring of $\mathcal{R}$

Grothendieck ring of $\mathcal{R}$

There is a ring isomorphism

$$\begin{aligned} \mathrm{Sym} &\to K_0(\mathcal{R}) \\ s_\lambda &\mapsto [S_\lambda] \end{aligned}$$

Example

- $e_n :=$ elementary symmetric functions
- $h_n :=$ complete symmetric functions

Then

$$\begin{aligned} \mathrm{Sym} &\to K_0(\mathcal{R}) \\ e_n &\mapsto \Bbbk_{sgn} \in \Bbbk\mathfrak{S}_n\mathrm{Mod} \\ h_n &\mapsto \Bbbk_{triv} \in \Bbbk\mathfrak{S}_n\mathrm{Mod} \end{aligned}$$

The Heisenberg ring Heis

Definition of Heis

For $f \in \mathrm{Sym}$:

- $f^+ : \mathrm{Sym} \to \mathrm{Sym}$ is given by left multiplication by f.
- $f^- : \mathrm{Sym} \to \mathrm{Sym}$ is the adjoint operator of f^+ with respect to the inner product on Sym: $(s_\lambda, s_\mu) = \delta_{\lambda,\mu}$.

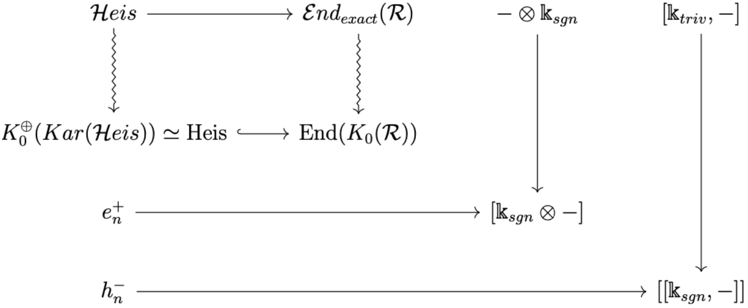
Heis **is the subring of** $\mathrm{End}(\mathrm{Sym})$ **generated by** f^+, f^- **for** $f \in \mathrm{Sym}$**.**

Properties of Heis

- Heis is generated by e_n^+, h_n^- for $n \in \mathbb{N}$.
- The action $\mathrm{Heis} \circlearrowright \mathrm{End}(K_0(\mathcal{R}))$ is defined by
  - $e_n^+ \cdot [X] = [\Bbbk_{sgn} \otimes X]$
  - $h_n^- \cdot [X] = [[\Bbbk_{triv}, X]]$

The Heisenberg category $\mathcal{H}eis$

Desideratum: We want a category $\mathcal{H}eis$ giving the situation:



What objects/morphisms should be in the image of $\mathcal{H}eis \to \mathcal{E}nd_{exact}(\mathcal{R})$:

- $-\otimes \Bbbk_{sgn}$ and $[\Bbbk_{triv}, -]$ are direct summands of $-\otimes \Bbbk\mathfrak{S}_n$ and $[\mathfrak{S}_n, -]$.
- Want the image to include objects $-\otimes \Bbbk\mathfrak{S}_n$, $[\mathfrak{S}_n, -]$ and their idempotent endomorphisms whose image is $-\otimes \Bbbk_{sgn}$ and $[\Bbbk_{triv}, -]$.

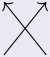
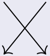
The Heisenberg category $\mathcal{H}eis$

Image of $\mathcal{H}eis \to \mathcal{E}nd_{exact}(\mathcal{R})$

- **Objects**: Identity $\mathbb{1}$ and compositions of

$$\uparrow := - \otimes \Bbbk\mathfrak{S}_1 : \mathcal{R} \to \mathcal{R}$$
$$\downarrow := [\Bbbk\mathfrak{S}_1, -] : \mathcal{R} \to \mathcal{R}$$

- **Morphisms**: Generated by

 $: - \otimes \mathfrak{S}_2 \to - \otimes \mathfrak{S}_2,$  $: [\mathfrak{S}_2, -] \to [\mathfrak{S}_2, -]$

and the unit/counit morphisms:

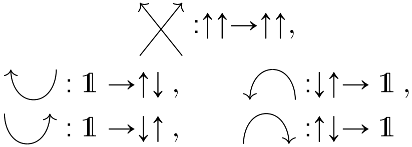
 $: \mathbb{1} \to \uparrow\downarrow,$  $: \downarrow\uparrow \to \mathbb{1},$

 $: \mathbb{1} \to \downarrow\uparrow,$  $: \uparrow\downarrow \to \mathbb{1}$

Khovanov's Heisenberg Category $\mathcal{H}eis$

Let $\Bbbk$ be a field of any characteristic. $\mathcal{H}eis$ is the strict monoidal $\Bbbk$-linear category with:

- **Objects**: Words in alphabet $\uparrow$, $\downarrow$. The empty word is denoted $\mathbb{1}$.
- **Generating Morphisms**:

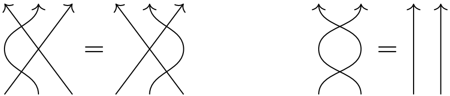


- **Composition of morphisms**: Vertical stacking
- **Tensor product of objects/morphisms**: Horizontal concatenation
- **Relations**: on next page

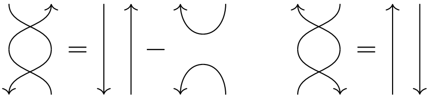
Khovanov's Heisenberg Category $\mathcal{H}eis$

Local relations:

(planar isotopy invariance)



(symmetric group relations)



(up-down crossings)

(circle/loop relations)

Khovanov and Brundan-Savage-Webster Result

If char $\Bbbk = 0$:

Recall the idempotents in $\Bbbk\mathfrak{S}_n$:

$$q_n = \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \sigma, \qquad q_n' = \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} (-1)^{\ell(\sigma)} \sigma$$

Khovanov [2014]: There is an embedding $\mathrm{Heis} \to K_0(\mathrm{Kar}(\mathcal{H}eis))$:

$$\begin{aligned} \mathrm{Heis} &\to K_0(\mathrm{Kar}(\mathcal{H}eis)) \\ e_n^+ &\mapsto [(\uparrow^{\otimes n}, q_n)] \\ h_n^- &\mapsto [(\uparrow^{\otimes n}, q_n')]. \end{aligned}$$

Brundan-Savage-Webster [2019]: This embedding is a ring isomorphism.

Categorical actions of $\mathcal{H}eis$

The functor $\mathcal{H}eis \to \mathcal{E}nd(\mathcal{R})$ is a special case of a **categorical action** of $\mathcal{H}eis$ on $\mathcal{R}$.

Categorical actions of $\mathcal{H}eis$

A categorical action of $\mathcal{H}eis$ on an abelian $\mathbb{k}$-linear category $\mathcal{C}$ is a monoidal functor

$$\Phi : \mathcal{H}eis \to \mathcal{E}nd(\mathcal{C})$$

where:

- $\mathcal{E}nd(\mathcal{C})$ is the category of endofunctors of $\mathcal{C}$.
- The monoidal product on $\mathcal{E}nd(\mathcal{C})$ is composition. e.g. so

$$\Phi(\uparrow \otimes \uparrow) = \Phi(\uparrow) \circ \Phi(\uparrow) : \mathcal{C} \to \mathcal{C}$$

Corollary of Khovanov-Brundan-Savage-Webster result

If $\mathcal{H}eis \circlearrowright \mathcal{C}$ then $\mathrm{Heis} = K_0(\mathrm{Kar}(\mathcal{H}eis)) \circlearrowright K_0(\mathcal{C})$

A characteristic-free approach

Hong-Yacobi [2013]

Over an algebraically closed field $\Bbbk$ of arbitrary characteristic:

- There is a categorical action $\mathcal{H}eis \circlearrowright \mathcal{P}$ on the category of polynomial functors.
- There are endofunctors $X_n, Y_n \in \mathcal{E}nd(\mathcal{P})$ that give a representation:

$$\begin{aligned} \mathrm{Heis} &\to \mathsf{End}(K_0(\mathcal{P})) \\ e_n^+ &\mapsto X_n \\ h_n^- &\mapsto Y_n \end{aligned}$$

- In characteristic zero this recovers the Heis action due to Khovanov's result.

A characteristic-free approach

How much of Khovanov's result can we extend to positive characteristic?

- **Problem**: The idempotents are not defined in positive characteristic.
- **Soulution**: Use coequalizers instead of idempotents.

Coequalizer functors

- Given a functor $F : \mathcal{C} \to \mathcal{C}$ and a finite family $X = \{\phi_1, \dots, \phi_n\}$ of endomorphisms of F, let $F_X : \mathcal{C} \to \mathcal{C}$ be the coequalizer of X in $\mathcal{E}nd(\mathcal{C})$.
- E.g. Consider the following subsets of $\Bbbk\mathfrak{S}_n$:

$$\mathfrak{S}_n^+ := \{\sigma \mid \sigma \in \mathfrak{S}_n\}, \qquad \mathfrak{S}_n^- := \{(-1)^{\ell(\sigma)}\sigma \mid \sigma \in \mathfrak{S}_n\}.$$

Then (in any characteristic):

$$- \otimes \Bbbk_{sgn} = (- \otimes \Bbbk\mathfrak{S}_n)_{\mathfrak{S}_n^-}, \qquad [\Bbbk_{triv}, -] = [\Bbbk\mathfrak{S}_n, -]_{\mathfrak{S}_n^+}$$

A characteristic-free approach

New Result (Hong-W.-Yacobi)

Given a categorical action $\Phi : \mathcal{H}eis \to \mathcal{E}nd(\mathcal{C})$, define the functors:

$$X_n := (\Phi(\uparrow^{\otimes n}))_{\mathfrak{S}_n^-}, \qquad Y_n := (\Phi(\downarrow^{\otimes n}))_{\mathfrak{S}_n^+}.$$

If all X_n, Y_n are exact then there is a representation $\mathrm{Heis} \to \mathsf{End}(K_0(\mathcal{C}))$ defined

$$e_n^+ \mapsto X_n$$
$$h_n^- \mapsto Y_n$$

Question

Under what conditions are the functors X_n and Y_n exact?

A characteristic-free approach

New Result (Hong-Yacobi-W.)

Suppose $\mathcal{C}$ is a closed symmetric monoidal $\Bbbk$-linear abelian category and $\mathcal{H}eis \circlearrowright \mathcal{C}$ via a map $\uparrow \mapsto - \otimes H$ for some object $H \in \mathcal{C}$. Then

$$X_n := - \otimes (H^{\otimes n})_{\mathfrak{S}_n^-} : \mathcal{C} \to \mathcal{C}, \qquad Y_n := [(H^{\otimes n})^{\mathfrak{S}_n^+}, -] : \mathcal{C} \to \mathcal{C}$$

and if

- $- \otimes -$ is exact in each variable.
- $\mathcal{C}$ has enough projectives and projective objects are closed under $\otimes$.
- $(H^{\otimes n})^{\mathfrak{S}_n^+}$ is projective.

Then X_n and Y_n are exact.

E.g. if char $\Bbbk = 0$ then $\mathcal{R}$ satisfies these conditions.
If char $\Bbbk \neq 0$ then $(\Bbbk\mathfrak{S}_n)^{\mathfrak{S}_n^+} = \Bbbk_{triv}$ is not projective and this criteria fails.

This criteria is satisfied by Hong-Yacobi's polynomial functors example.

A characteristic-free approach

Prototypical examples of actions $\mathcal{H}eis \circlearrowright \mathcal{C}$ have the following structure:

- $\mathcal{C}$ is a closed symmetric monoidal $\Bbbk$-linear abelian category.
  Recall: closed means $\mathcal{C}$ has an internal hom $[-,-] : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{C}$.
- $\Phi(\uparrow) = - \otimes H$ for some object $H \in \mathcal{C}$.
- By adjunction/planar-isotopy $\Phi(\downarrow) = [H, -]$ for some object $H \in \mathcal{C}$.
- The crossings are given by the symmetric braiding isomorphisms for $\otimes$.

Further Generalizations

For $k \in \mathbb{Z}$ there are:

- Rings Heis_k
- Pivotal $\Bbbk$-linear categories $\mathcal{H}eis_k$

where $\mathrm{Heis} = \mathrm{Heis}_{-1}$ and $\mathcal{H}eis = \mathcal{H}eis_{-1}$.

Brundan-Savage-Webster [2019]

$K_0(\mathcal{H}eis_k) \simeq \mathrm{Heis}_k$ for all $k \in \mathbb{Z}$.

Conjecture

Let $\Bbbk$ be a field of any characteristic. Any categorical action $\mathcal{H}eis_k \circlearrowright \mathcal{C}$ gives rise to a ring action $\mathrm{Heis}_k \circlearrowright K_0(\mathcal{C})$.

New result (Hong-W.-Yacobi)

This conjecture is true when $|k| \leqslant 2$.

Further Generalizations

Khovanov-Licata-Savage category

Our results also hold in Licata-Savage's [2013] quantization of $\mathcal{H}eis$.

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